A geometrical insight on pseudoconvexity and pseudomonotonicity

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Abstract

Generalised convexity is revisited from a geometrical point of view. A substitute to the subdifferential is proposed. Then generalised monotonicity is considered. A representation of generalised monotone maps allows to obtain a symmetry between maps and their inverses. Finally, maximality of generalised monotone maps is analysed.

Keywords: Generalised convexity, generalised monotonicity, maximality, cyclic properties, revealed preferences axioms

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1 Introduction

A general formulation of a global optimisation problem in $\mathbb{R}^n$ is:

$$\text{Find } x \in C \text{ such that } f(x) \leq f(y) \text{ for all } y \in C$$

where $f : \mathbb{R}^n \to [-\infty, +\infty]$ and $\emptyset \neq C \subseteq \mathbb{R}^n$. The problem is equivalent to

$$\text{Find } x \in C \text{ such that } C \cap \widetilde{S}(x) = \emptyset \text{ where } \widetilde{S}(x) = \{y \in X \,:\, f(y) < f(x)\}. \quad (1)$$

When C is convex and $\widetilde{S}(x)$ is open, convex and nonempty, the problem is still equivalent to

$$\text{Find } x \in C,\ x^* \in \mathbb{R}^n \text{ st } \langle x^*, y - x\rangle < 0 \leq \langle x^*, z - x\rangle \quad \forall\, y \in \widetilde{S}(x),\ \forall\, z \in C, \quad (2)$$

and, if in addition x belongs to the boundary of $\widetilde{S}(x)$,

$$\text{Find } x \in C,\ x^* \in \widetilde{N}(x),\ x^* \neq 0 \text{ such that } 0 \leq \langle x^*, z - x\rangle \quad \forall\, z \in C \quad (3)$$

where

$$\widetilde{N}(x) = \{\, x^* \,:\, \langle x^*, y - x\rangle \leq 0 \quad \forall\, y \in \widetilde{S}(x)\,\}.$$

One of the main objects of sensitivity analysis consists in the analysis of the behaviour of the set of optimal solutions of the problem when C is subject to small perturbations. This motivates the analysis of the behaviour of the normal cones $\widetilde{N}(x)$ and is but one of the purposes of this paper.

A function with convex lower level sets is called quasiconvex, but the class of these functions is too large with respect to necessary and sufficient conditions for optimality and it is necessary to restrict the class. The class of pseudoconvex functions is a good response but it applies only to differentiable functions. We enlarge this class to nondifferentiable functions using an approach based on the geometry of the level sets instead of the usual analytic approaches based on generalised derivatives.

The geometric structure of convex sets is indeed the good approach, even for convex functions. By definition f is convex if and only if its epigraph is convex. Let f be a proper convex function. Given $a \in \operatorname{int}(\operatorname{dom}(\mathrm{f}))$ let us denote by $E(a)$ the normal cone at $(a, f(a))$ to the epigraph of f:

$$E(a) = \{(a^*, \lambda^*) \,:\, \langle a^*, x - a\rangle + \lambda^*(\lambda - f(a)) \leq 0 \ \ \forall\, (x, \lambda) \in \operatorname{epi}(\mathrm{f})\}.$$

The geometric (and, in our opinion, it is more meaningful than the classical analytic) definition of the subdifferential of f at a is:

$$a^* \in \partial f(a) \Longleftrightarrow (a^*, -1) \in E(a).$$

Then $E(a)$ is the closed convex cone generated by the nonempty compact convex set $\partial f(a) \times \{-1\}$. If $f(a) > \inf f$, the normal cone to the level set $\widetilde{S}(a)$ is generated by $\partial f(a)$. Sensitivity analysis in convex programming is classically performed via the subdifferentials $\partial f(x)$ whose the properties are nothing else that direct consequences of the properties of the normal cones $E(x)$ to the epigraph of f and therefore of the convexity of the epigraph. In the same manner, the properties of the directional derivatives $f'(.,.)$,

$$f'(x,d) = \sup[\langle x^*, d\rangle \ : \ x^* \in \partial f(x)\,]$$

are nothing else that secondary consequences of the properties of the normal cone to a convex set.

The convexity structure of a quasiconvex function on $\mathbb{R}^n$ is on its level sets (subsets of $\mathbb{R}^n$) and not on its epigraph (a subset of $\mathbb{R}^{n+1}$). Because this loss of one dimension, the normal cones to the level sets cannot be deduced from the normal cone to the epigraph and important properties as the continuity, the local Lipshitz condition, the existence of directional derivatives, disappear. In fact, they are no really appropriate generalised derivatives, generalised subdifferentials applicable to quasiconvex functions. All attempts, and they have been many, made to define such objects are fruitless because without links with the essence of quasiconvexity, the convexity of the level sets. The reason is that in the convex case we work on the unique function f associated to the epigraph while in the quasiconvex case we work not one specific function but on the class of all functions sharing the same level sets $\tilde{S}(x)$. Given one function f in the class, other functions are the results of the composition of an increasing function k with f, i.e., they are of the form $k \circ f$. No continuity, differentiability properties are required on k. No generalised derivatives are associated to the class.

We shall show that the normal cones to the level sets of quasiconvex functions enjoy many rich properties. Still, the fact that they contain 0 and are unbounded brings some difficulties in their use. These difficulties are overcome in the convex case by the use of the subdifferential: if f is a proper closed convex function which does not reach its minimum at a point a in the interior of the domain of f, then $\partial f(a)$ is a compact convex nonempty generator of $\widetilde{N}(a)$ and $0 \notin \partial f(a)$. For a special class

of quasiconvex functions, called the class of g-pseudoconvex functions, we introduce in section 3 similar generators of the normal cones , they are convex compact and they do not contain 0. They can be considered as substitutes of the subdifferential, their use makes easier the analysis of the behaviour of the map normal cone.

Convexity is closely related to monotonicity. Similarly, quasiconvexity is closely related to quasimonotonicity. Still, as in the case of functions where convexity is proper to a specific function and quasiconvexity to a class of functions, monotonicity is proper to a specific monotone map, and quasimonotonicity to a class of equivalent maps, Σ and Γ being said equivalent if for all x

$$\{\lambda x^* \,:\, \lambda > 0,\ x^* \in \Gamma(x)\} = \{\mu y^* \,:\, \mu > 0,\ y^* \in \Sigma(x)\}.$$

It is convenient to choose a good representation of the class by a particular map Γ. The substitutes to the subdifferential introduced in section 3 would be convenient if one property of symmetry was not missing. Indeed, if Γ is monotone, so its inverse map Γ^{-1} is. This property does not hold for the maps Γ_e of section 3. This is the reason of section 5 where we introduce a map representing the class leading to the wished property.

Maximality is of major importance in the study of monotone maps. In particular, if the interior of the domain of a maximal monotone map is nonempty, then this interior is convex and its closure coincides with the closure of the domain. It is known that for any monotone map Γ, there is a (not necessarily unique) maximal monotone map Σ containing Γ. In case where there are $S \subseteq \mathrm{dom}\,(\Gamma)$ and a neighbourhood V of a with $\mathrm{cl}\,(\mathrm{V}) = \mathrm{cl}\,(\mathrm{V} \cap \mathrm{S}$, then $\Sigma(a)$ is uniquely defined. This property is quite significant because it said that for a convex function f, $\partial f(a)$ is thoroughly defined from the gradients of f at points in the neighbourhood of a where f is differentiable. In Section 6, we show that quite similar properties hold for pseudomonotone maps.

2 Quasiconvexity versus pseudoconvexity

The formulations of problems (2) and (3) justify the introduction of quasiconvex functions in the following unconventional way.

A function $f : \mathbb{R}^n \to (-\infty, +\infty]$ is said to be *quasiconvex* if for each $x \in \mathbb{R}^n$ the strict lower level set

$$\widetilde{S}(x) = \{y \in \mathbb{R}^n \,:\, f(y) < f(x)\}$$

is convex. Then, for any $\lambda \in \mathbb{R}$, the level set

$$\widetilde{S}_\lambda = \{x \in \mathbb{R}^n \,:\, f(x) < \lambda\}$$

is convex. Indeed, if this set is not convex, there are $x, y \in \widetilde{S}_\lambda$ and $t \in (0,1)$ such that $f(z = tx + y - ty) \geq \lambda$. Then x and y are in $\widetilde{S}(z)$ and z is not in contradiction with the convexity of $\widetilde{S}(z)$. It follows that the sets

$$\begin{aligned} S_\lambda &= \{x \in \mathbb{R}^n \,:\, f(x) \leq \lambda\} = \cap_{\mu > \lambda} \widetilde{S}_\mu, \\ S(x) &= \{y \in \mathbb{R}^n \,:\, f(y) \leq f(x)\} \end{aligned}$$

are convex. This is also the case of

$$\mathrm{dom\,(f)} = \{\mathrm{x} \in \mathbb{R}^{\mathrm{n}} \,:\, \mathrm{f(x)} < +\infty\}.$$

An equivalent analytic definition is: f is quasiconvex if and only if

$$f(tx + (1-t)y) \leq \max\,[\, f(x), f(y)\,] \quad \forall\, x, y \in \mathbb{R}^n,\ \forall\, t \in [0,1]. \tag{4}$$

As usual in convex analysis, given $D \subset \mathbb{R}^n$, a function $f : D \to \mathbb{R}$ is extended to the whole space $\mathbb{R}^n$ in taking $f(x) = +\infty$ if $x \notin D$. f is said to be quasiconvex on D if its extension is quasiconvex.

If $D \subseteq \mathbb{R}^n$ is convex and $f : D \to \mathbb{R}$ is differentiable, f is quasiconvex on D if and only if

$$x, y \in D,\ f(y) < f(x) \Longrightarrow \langle \nabla f(x), y - x \rangle \leq 0. \tag{5}$$

The main defect of quasiconvex functions is that a local minimum is not necessarily global as shown by the function of one real variable defined by $f(x) = x + 1$ if $x \leq -1$, $f(x) = 0$ if $0 \leq x \leq 1$ and $f(x) = x - 1$ if $x \geq 1$.

Another defect of quasiconvex functions with respect to optimality conditions is that $\nabla f(x) = 0$ does not imply that x reaches a minimum at x as shown by the function of one real variable $f(x) = x^3$. Pseudoconvex functions have been introduced to remedy these failures: given D convex and $f : D \to \mathbb{R}$ differentiable, f is said to be *d-pseudoconvex* on D if

$$x, y \in D,\ f(y) < f(x) \Longrightarrow \langle \nabla f(x), y - x \rangle < 0. \tag{6}$$

The letter "d" has been added to pseudoconvex in order to emphasise the fact that the definition applies only to differentiable functions. One notes that if C is convex

and f is d-pseudoconvex on $D \supseteq C$, a necessary and sufficient condition in order that f reaches its minimum at $x \in C$ on C is

$$\langle \nabla f(x), z - x \rangle \geq 0 \quad \forall\, z \in C$$

to be compared with (3). One notes that$\nabla f(x) \in \widetilde{N}(x)$ when $f(x) > \inf f$.

Convex differentiable functions are d-pseudoconvex and d-pseudoconvex functions are quasiconvex. We have the following proposition.

Proposition 2.1 *Assume that D is open and convex and f is continuously differentiable on D. Then f is d-pseudoconvex on D if and only if f is quasiconvex on D and has a local minimum at any $x \in D$ such that $\nabla f(x) = 0$.*

They have been many attempts to extend pseudoconvexity to nondifferentiable functions. They are based in general on generalised derivatives or subdifferentials. But they are no really appropriate generalised subdifferentials for quasiconvex functions, these functions have no Lipshitz properties, even more they are in general not continuous in the interiors of their domains. Actually, a pure analytic approach neglects the essential feature of quasiconvexity: the convexity of the level sets. Our approach, based on the geometry of the level sets, is motivated by conditions (2) and (3).

We say that $f : \mathbb{R}^n \to [-\infty, \infty]$ is *g-pseudoconvex* if quasiconvex and for any x with $f(x) > \inf f$ one has

$$\emptyset \neq \operatorname{int}(\mathrm{S(x)}) \subseteq \widetilde{\mathrm{S}}(\mathrm{x}) \ \text{ and } \ \mathrm{S(x)} \subseteq \operatorname{cl}(\widetilde{\mathrm{S}}(\mathrm{x})).$$

Thus $S(x)$ and $\widetilde{S}(x)$ share the same interior and the same closure. The letter "g" indicates the geometric aspect of the definition. A g-pseudoconvex function is not continuous in general (consider the function of one real variable $f(x) = x-1$ if $x < 0$, $f(0) = 0$ and $f(x) = x + 1$ if $x > 0$). A proper convex function is g-pseudoconvex. Any local minimum of a g-pseudoconvex function is global. A differentiable g-pseudoconvex function is not necessarily d-pseudoconvex as shown by the function of one real variable $f(x) = x^3$.

Let us turn our interest in the normal cones to the level sets of a function. We start with a general property where no generalised convexity type assumption is required.

Proposition 2.2 *Assume that $f : \mathbb{R}^n \to [-\infty, \infty]$ is lower semi-continuous at x and $f(x) > \inf f$. Then $\widetilde{N}$ is closed at x, i.e., if the sequence $\{x_k\}$ converges to x, the sequence $\{x_k^*\}$ converges to x^* and $x_k^* \in \widetilde{N}(x_k)$ for all k, then $x^* \in \widetilde{N}(x)$.*

Proof: We must prove that for any y with $f(y) < f(x)$ one has $\langle x^*, y - x\rangle \leq 0$. For k large enough one has $f(y) < f(x_k)$ and thereby $\langle x_k^*, y - x_k\rangle \leq 0$. Now pass to the limit as k goes to ∞ to obtain the result. ■

By convention, we take $\widetilde{N}(x) = \mathbb{R}^n$ when $f(x) = \inf f(y)$. Then $\widetilde{N}$ is closed on the whole space $\mathbb{R}^n$ when f is lower semi-continuous.

Let us consider now some specific properties of g-pseudoconvex functions. Let us introduce

$$N(x) = \{\, x^* \,:\, \langle x^*, y - x\rangle \leq 0 \quad \forall\, y \in S(x)\,\}.$$

One important consequence of the definition of g-pseudoconvexity is that for any x with $f(x) > \inf f$ one has $N(x) = \widetilde{N}(x)$ when f is g-pseudoconvex.

Assume now that f is g-pseudoconvex continuous on the convex set C and x, y are in C with $f(x) > \inf f$. Then

$$x^* \in N(x) \text{ and } f(y) \leq f(x) \implies \langle x^*, y - x\rangle \leq 0, \tag{7}$$

$$0 \neq x^* \in N(x) \text{ and } f(y) < f(x) \implies \langle x^*, y - x\rangle < 0. \tag{8}$$

Or equivalently,

$$x^* \in N(x) \text{ and } \langle x^*, y - x\rangle > 0 \implies f(y) > f(x), \tag{9}$$

$$0 \neq x^* \in N(x), \text{ and } \langle x^*, y - x\rangle \geq 0 \implies f(y) \geq f(x). \tag{10}$$

Let us introduce the point-to-set map N^* defined by

$$N^*(x) = \begin{cases} \{0\} & \text{if} \quad f(x) = \inf f \\ \{x^* \in N(x),\, x^* \neq 0\} & \text{if} \quad f(x) > \inf f. \end{cases}$$

Then

$$x^* \in N^*(x) \text{ and } \langle x^*, y - x\rangle \geq 0 \implies f(y) \geq f(x). \tag{11}$$

Assume now that for $i = 0, 1, \cdots, p-1$ it holds

$$x_i \in C, \quad x_i^* \in N^*(x_i) \text{ and } \langle x_i^*, x_{i+1} - x_i\rangle \geq 0.$$

Then (11) implies $f(x_0) \leq f(x_1) \leq \cdots \leq f(x_p)$. Next, (7) implies $\langle x_p^*, x_0 - x_p\rangle \leq 0$. Furthermore, if one of the inequalities $\langle x_i^*, x_{i+1} - x_i\rangle \geq 0$ is strict, (9) implies $f(x_0) < f(x_p)$ and (8) implies $\langle x_p^*, x_0 - x_p\rangle < 0$. In other words, $\langle x_p^*, x_0 - x_p\rangle = 0$ implies $\langle x_i^*, x_{i+1} - x_i\rangle = 0$ for $i = 0, 1, \cdots, p-1$. We resume these observations in the following proposition which corresponds to the cyclic monotonicity property of convex functions.

Proposition 2.3 *Assume that f is g-pseudoconvex continuous on C convex and we are given a finite family $\{(x_i, x_i^*)\}_{i=0,\cdots,p+1} \subset \mathrm{graph}\,(\mathrm{N}^*)$ such that $(x_0, x_0^*) = (x_{p+1}, x_{p+1}^*)$ and, for $i = 0, 1, \cdots, p$, $x_i \in C$. Then, one has*

$$\min_{i=0,\cdots,p} \langle x_i^*, x_{i+1} - x_i \rangle \leq 0$$

and, if the minimum is zero, $\langle x_i^*, x_{i+1} - x_i \rangle = 0$ for $i = 0, \cdots, p$.

This property will be analysed in a more general way in section 4.

3 Generate the normal cone, first manner

Sensitivity analysis in convex programming is based on subdifferentials. This section introduces a substitute for the subdifferential appropriate to quasi/pseudoconvex functions. It is based on a local monotonicity of these functions.

Before, let us recall that the positive polar cone of $D \subset \mathbb{R}^n$, is defined as

$$D^+ = \{x^* \in \mathbb{R}^n \,:\, \langle x^*, x \rangle \geq 0 \quad \forall\, x \in D\}.$$

If D is an open nonempty convex cone, the relative interior of D^+ is the convex cone

$$\mathrm{ri}\,(\mathrm{D}^+) = \{\mathrm{x}^* \in \mathbb{R}^{\mathrm{n}} \,:\, \langle \mathrm{x}^*, \mathrm{x} \rangle > 0 \quad \forall\, \mathrm{x} \in \mathrm{D}\}.$$

Proposition 3.1 *Assume that f is quasiconvex and $a \in \mathrm{int}\,(\mathrm{dom}\,(\mathrm{f}))$ is such that there exists λ with $\mathrm{int}\,(\mathrm{S}_\lambda) \neq \emptyset$ and $a \notin \mathrm{cl}\,(\mathrm{S}_\lambda)$. Then there exist a convex open neighbourhood $V \subseteq \mathrm{dom}\,(\mathrm{f})$ of a and an open nonempty convex cone D such that $f(y) \leq f(x)$ whenever $x, y \in V$ with $x - y \in D$.*

Assume in addition that f is g-pseudoconvex. Then $f(y) < f(x)$ whenever $x, y \in V$ with $x - y \in D$. Fix some $e \in D$, then $\langle x^*, e \rangle > 0$ for all $0 \neq x^* \in N(x)$ with $x \in V$. It follows that $\{0\} \neq N(x) = \widetilde{N}(x) \subseteq \mathrm{ri}\,(\mathrm{D}^+) \cup \{0\}$ whenever $x \in V$.

Proof: Choose some $b \in \mathrm{int}\,(\mathrm{S}_\lambda)$ and some $\bar{t} > 0$ such that $c = a + \bar{t}(a - b) \in \mathrm{dom}\,(\mathrm{f})$. Consequently, $c \notin \mathrm{cl}\,(\mathrm{S}_\lambda)$. Define

$$D = \{d = \xi(c - z) \,:\, \xi > 0,\, z \in \mathrm{int}\,(\mathrm{S}_\lambda)\}.$$

Then D is an open convex nonempty cone and $a \in c - D$. Because the assumptions, there exists an open convex neighbourhood $V \subseteq c - D$ of a such that $V \cap \mathrm{cl}\,(\mathrm{S}_\lambda) = \emptyset$.

Let $x \in V$ and $d \in D$ such that $y = x - d \in V$. By assumption there exist $t > 0$ such that $z_1 = c - td \in \text{int}\,(\mathrm{S}_\lambda)$, $s \in (0,1)$ and $z_2 \in \text{int}\,(\mathrm{S}_\lambda)$ such that $x = sc + (1-s)z_2$. Hence $x = sz_1 + (1-s)z_1 + tsd$. It results that $z = x - tsd = x + ts(y-x) \in \text{int}\,(\mathrm{S}_\lambda)$. Since y and x are in the convex set V and $z \notin V$, $ts \leq 1$ cannot occur. Then,

$$y = x + \frac{1}{ts}(z - x),\ f(x) > \lambda \geq f(z),\ \ 1 < ts.$$

It follows that that $f(y) \leq f(x)$. For y' close to y one has also $f(y') \leq f(x)$ and therefore $y \in \text{int}\,(\mathrm{S(x)})$ which implies, when f is g-pseudoconvex, $y \in \widetilde{S}(x)$. The proofs of the other claims are immediate. ■

Assume that f is convex and $a \in \text{int}\,(\text{dom}\,(\mathrm{f}))$ with $f(a) > \inf f$. Then

$$\widetilde{N}(a) = N(a) = \{\lambda a^* \,:\, \lambda \geq 0,\ \ a^* \in \partial f(a)\}.$$

where $\partial f(a)$ denotes the subdifferential of f at a. The fact that the normal cone is generated by a convex compact set which does not contain the origin makes easier the sensitivity analysis than in dealing directly with the normal cone. We shall propose a substitute for $\partial f(a)$ when f is g-pseudoconvex.

Assume that f is a lower semi-continuous g-pseudoconvex function and $a \in \text{int}\,(\text{dom(f)})$ with $f(a) > \inf f$. Let V and D be defined as in proposition 3.1. Fix some $e \in D$ and define $T \subset \mathbb{R}^n$ and the point-to-set map $\Gamma_e : V \rightrightarrows \mathbb{R}^n$ by

$$\Gamma_e(x) = \{x^* \in N(x) \,:\, \langle x^*, e\rangle = 1\} \subseteq \{x^* \in D^+ \,:\, \langle x^*, e\rangle = 1\} = T. \qquad (12)$$

By construction, T is closed, convex and bounded. We list below some properties of Γ_e.

Proposition 3.2 *For all $x \in V$ it holds,*

1. $N(x) = \{\lambda x^* \,:\, \lambda \geq 0,\ \ x^* \in \Gamma_e(x)\}$.
2. $\Gamma_e(x)$ *is nonempty, bounded, closed and convex.*
3. *The point-to-set map Γ_e is closed at x.*
4. *For any $\Omega \supset \Gamma_e(x)$, there exists a neighbourhood $W \subseteq V$ of x such that $\Omega \supset \Gamma_e(y)$ for all $y \in W$.*

Proof: Items 1 and 2 are immediate consequence of definitions. Γ_e is closed at x since N is so. Item 4 is a consequence of item 3 and the inclusion $\Gamma_e(x) \subseteq T$. ■

Next, given $x \in V$ and $h \in \mathbb{R}^n$, let us define

$$\theta_e^+(x,h) = \sup[\,\langle x^*, h\rangle \,:\, x^* \in \Gamma_e(x)\,], \tag{13}$$

$$\theta_e^-(x,h) = \inf[\,\langle x^*, h\rangle \,:\, x^* \in \Gamma_e(x)\,]. \tag{14}$$

Proposition 3.3 *Let $\bar{x} \in V$, $\bar{h} \in \mathbb{R}^n$ and $\varepsilon > 0$ be given. Then, there exist $\alpha > 0$ such that for all $x \in V$ and $h \in \mathbb{R}^n$ with $\|x - \bar{x}\| < \alpha$ and $\|h - \bar{h}\| < \alpha$ one has*

$$\theta_e^-(\bar{x},\bar{h}) - \varepsilon \leq \theta_e^-(x,h) \leq \theta_e^+(x,h) \leq \theta_e^+(\bar{x},\bar{h}) + \varepsilon$$

Proof: Assume that the inequality on the right does not hold. Then, there exist a sequence $\{(x_k,h_k)\}$ converging to $(\bar{x},\bar{h})$ with $\theta_e^+(x_k,h_k) > \theta_e^+(\bar{x},\bar{h}) + \varepsilon$. Let $x_k^* \in \Gamma_e(x_k)$ be such that $\theta_e^+(x_k,h_k) = \langle x_k^*, h_k\rangle$. The sequence $\{x_k^*\}$ being included in the compact set T has cluster points. Let $\bar{x}^*$ be such a cluster point, it belongs to $\Gamma_e(\bar{x})$. Passing to the limit, one has the contradiction: $\theta_e^+(\bar{x},\bar{h}) \geq \langle \bar{x}^*, \bar{h}\rangle \geq \theta_e^+(\bar{x},\bar{h}) + \varepsilon$. The inequality on the left is proved in the same way. ■

But the closure property is a weak form of continuity for point-to-set maps as illustrated via the map Σ defined on $\mathbb{R}$ by $\Sigma(x) = \{0\}$ if $x \neq 0$ and $\Sigma(0) = [-1,1]$. Stronger forms apply to subdifferentials of convex functions, indeed given f convex, $a \in \mathrm{int}\,(\mathrm{dom}\,(\mathrm{f}))$ and S locally dense in a neighbourhood of a, one knows that $\partial f(a)$ is the closed convex hull of the cluster points of sequences $\{x_k^*\}$ where $(x_k, x_k^*) \in \mathrm{graph}\,(\partial \mathrm{f})$ and the sequence $\{x_k\} \subset S$ converges to a. We shall state a similar property for pseudoconvex functions.

Let us associate with $N(a)$ and $S \subset \mathbb{R}^n$ the cone $N_l(a,S)$ defined as follows: $a^* \in N_l(a,S)$ if there exists a sequence $\{(x_k, x_k^*)\} \subset \mathrm{graph}\,(\mathrm{N}) \cap \mathrm{S} \times \mathbb{R}^{\mathrm{n}}$ converging to (a,a^*). Next, let us define $N_c(a,S)$ as the closed convex hull of $N_l(a,S)$. In the same manner we define $\Gamma_{el}(a,S)$ and $\Gamma_{ec}(a,S)$ from $\Gamma_e(a)$.

Theorem 3.1 *Assume that f is a lower semi-continuous g-pseudoconvex function and $a \in \mathrm{int}\,(\mathrm{dom}\,(\mathrm{f}))$ with $f(a) > \inf f$. Let V and D be defined as in proposition 3.1, let $e \in D$. Next let $S \subseteq V$ be such that $\mathrm{cl}\,(\mathrm{V}) = \mathrm{cl}\,(\mathrm{V} \cap \mathrm{S})$. Then $N(a)$ coincides with $N_c(a,S)$ and $\Gamma_e(a)$ coincides with $\Gamma_{ec}(a,S)$.*

Proof: i) Since $N(a)$ is closed and convex, it follows from proposition 2.2 that $N(a)$ contains $N_l(a,S)$ and therefore $N_c(a,S)$. Let us prove the reverse inclusion.

It is enough to prove that if a^* belongs to the relative boundary of $\Gamma_e(a)$, there is a sequence $\{(a_k, a_k^*)\} \subset \text{graph}\,(\text{N}) \cap \text{S} \times \mathbb{R}^{\text{n}}$ converging to (a, a^*). Without loss of generality we assume that e is the n-th vector of the canonical basis of $\mathbb{R}^n$. Given $x \in \mathbb{R}^n$, we write $x = (y, \psi)$ where $y \in \mathbb{R}^{n-1}$ corresponds to the $(n-1)$ first entries of x and ψ is the last one. In particular, $a = (b, \beta)$. In a neighbourhood of a the boundary of the convex level set $S(a) = \{x : f(x) \leq f(a)\}$ can be described via a convex function g in the following manner: (y, ψ) belongs to the boundary if and only if $\psi = -g(y)$. In particular $g(b) = -\beta$ and $N(a)$ is the closed convex cone generated by the convex set $(\partial g(b), 1)$ where ∂g denotes the subdifferential of the convex function g.

ii) Let $a^* = (b^*, 1)$ in the relative boundary of $\Gamma_e(a)$. Then b^* is in the relative boundary of $\partial g(b)$. For any positive integer k, there exists some y_k such that $(y_k, -g(y_k)) \in V$, $\|y_k - b\| < k^{-1}$, g is differentiable at y_k and $\|\nabla g(y_k) - b^*\| < k^{-1}$. Since N is closed at the point $x_k = (y_k, -g(y_k))$ and $N(x_k)$ is the cone generated by the singleton $\{x_k^* = (\nabla g(y_k), 1)\}$, there exists $a_k \in S$ and $a_k^* \in N(a_k)$ such that $\|x_k - a_k\| < k^{-1}$ and $\|x_k^* - a_k^*\| < k^{-1}$. Then, $\|a_k^* - a^*\| < 2k^{-1}$ and $\|a_k - a\| < 2k^{-1}$. The remaining of the proof is immediate. ■

The following result on the continuity of normal cones will be used later.

Theorem 3.2 *Assume that f is is a lower semi-continuous g-pseudoconvex function on the convex set C. Assume that $a \in \text{int}\,(\text{C})$ with $f(a) > \inf f$. Assume that $b \in C$ and there exists some $a^* \in N(a)$ such that $\langle a^*, b - a\rangle > 0$. Then there is $\bar{t} \in (0, 1)$ such that for any $t \in (0, \bar{t})$ there exists a neighbourhood W_t of $a + t(b - a)$ such that*

$$\langle x^*, b - x\rangle > 0 \quad \forall\, x \in W_t, \;\; \forall\, x^* \in N(x),\, x^* \neq 0.$$

Proof: Let V, D, e, Γ_e, θ_e^+ and θ_e^- be defined as in (12), (13) and (14). Set $h = b - a$. We choose $\bar{t} \in (0, 1)$ so that $a + \bar{t}h \in V$. Let any $t \in (0, \bar{t}\,)$. The assumption implies $\langle a^*, a + th - a\rangle > 0$. Hence, since f is g-pseudoconvex, one has $\langle x^*, a + th - a\rangle > 0$ for all $x^* \in \Gamma_e(a + th)$ with $x^* \neq 0$ and therefore $\theta_e^-(a + th, h) > 0$. From proposition 3.3 we deduce that $\alpha > 0$ exists such that $x = a + th + (1 - t)y \in V$ and $\theta_e^-(x, h - y) > 0$ whenever $\|y\| < \alpha$. On the other hand, $b - x = (1 - t)(h - y)$ and therefore $\langle x^*, b - x\rangle > 0$ for all $x^* \in \Gamma_e(x)$. The result follows. ■

4 Generalised monotonicity

A point-to-set map $\Sigma : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ is said to be *quasimonotone of order p* if for all $(x_i, x_i^*) \in \text{graph}\,(\Sigma)$, $\text{i} = 0, 1, \cdots, \text{p} + 1$, with $(x_0, x_0^*) = (x_{p+1}, x_{p+1}^*)$, one has

$$\min_{i=0,\cdots,p} \langle x_i^*, x_{i+1} - x_i \rangle \leq 0.$$

$\Sigma : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ is said to be *pseudomonotone of order p* if in addition one has $\langle x_i^*, x_{i+1} - x_i \rangle = 0$ for all i when the minimum is zero.

Σ is said to be *cyclically quasimonotone* (*cyclically pseudomonotone*) if quasimonotone (pseudomonotone) of any order p.

Σ is said to be *quasimonotone* (*pseudomonotone*) if quasimonotone (pseudomonotone) of order 0. Thus Σ is quasimonotone if for all $(x, x^*), (y, y^*) \in \text{graph}\,(\Sigma)$

$$\langle x^*, y - x \rangle > 0 \Longrightarrow \langle y^*, y - x \rangle \geq 0,$$

and pseudomonotone if

$$\langle x^*, y - x \rangle > 0 \Longrightarrow \langle y^*, y - x \rangle > 0.$$

A (cyclically) monotone map is (cyclically) pseudomonotone. A pseudomonotone map is quasimonotone. The gradient of a differentiable quasiconvex (d-pseudoconvex) function is cyclically quasimonotone (cyclically pseudomonotone). Proposition 2.3 justifies the introduction of our geometric concept of pseudoconvexity, indeed, the map N^* defined from the normal cones of a g-pseudomonotone continuous function is cyclically pseudomonotone. It is the case also of the map Γ_e introduced in the last section.

The following result is a transposition of proposition 2.1 from functions to single-valued maps. For completeness we recall the proof.

Proposition 4.1 *Assume that $\Sigma : D \to \mathbb{R}^n$ is continuous and does not vanish on the open convex set $D \subseteq \mathbb{R}^n$. Then Σ is pseudomonotone on D if and only if it is quasimonotone on D.*

Proof: Assume for contradiction that Σ is quasimonotone but not pseudomonotone. There are $a, b \in C$, such that $\langle \Sigma(a), b - a \rangle > 0$ and $\langle \Sigma(b), a - b \rangle = 0$. Since Σ is continuous, $\langle \Sigma(a + t\Sigma(b)), b - a - t\Sigma(b) \rangle > 0$ for $t > 0$ small enough. On the other hand $\langle \Sigma(b), a + t\Sigma(b) - b \rangle = t\|\Sigma(b)\|^2 > 0$ in contradiction with Σ quasimonotone. ■

The continuity plays an important role in the proof. In consequence the extension to multi-valued maps requires also a continuity condition.

We say that Σ is *c-continuous* at $a \in \mathrm{dom}\,(\Sigma)$ if for all $\bar{t} > 0$, $b \in \mathbb{R}^n$ and $a^* \in \Sigma(a)$ such that $\langle a^*, b-a\rangle > 0$, there exists $t \in (0, \bar{t}\,)$ and a neighbourhood $W_t \subset \mathrm{dom}\,(\Sigma)$ of $a + t(b-a)$ such that for all $x \in W_t$ there exists $x^* \in \Sigma(x)$ with $\langle x^*, b-x\rangle > 0$.

Because $\bar{t} > 0$ can be taken as small as wanted, the terminology continuous is quite appropriate. For single-valued maps, c-continuity is weaker that the usual continuity. The subdifferential of a proper convex function is c-continuous at any point in the interior of its domain. With respect to the object of our study, the map N^* in theorem 3.2 is c-continuous.

Theorem 4.1 *Let $\Sigma : C \longrightarrow \mathbb{R}^n$ with $\emptyset \neq C \subseteq \mathbb{R}^n$ open and convex. Assume that $0 \notin \Sigma(x) \neq \emptyset$ for all $x \in C$. Assume in addition that Σ is c-continuous on C. Then Σ is pseudomonotone on C if and only if it is quasimonotone on C.*

Proof: If Σ is quasimonotone and not pseudomonotone there are $a, b \in C$, $a^* \in \Sigma(a)$ and $b^* \in \Sigma(b)$ such that $\langle a^*, b-a\rangle > 0$ and $\langle b^*, b-a\rangle = 0$. Take $t > 0$ as in the continuity condition and $x = a + t(b-a) + sb^*$ with $s > 0$ small enough so that $x \in W_t$. Then $\langle b^*, x-b\rangle = s\|b^*\|^2 > 0$ in contradiction with Σ quasimonotone. ■

With $\Sigma : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ let us associate $\widehat{\Sigma}$ by

$$\widehat{\Sigma}(x) = \cup_{\lambda>0}\lambda\Sigma(x) \quad \forall\, x \in \mathbb{R}^n.$$

Then Σ is quasimonotone, pseudomonotone, cyclically quasimonotone, cyclically pseudomonotone if and only $\widehat{\Sigma}$ is so. Thus these concepts of generalised monotone apply essentially not to a specific Σ but to the class of maps Σ generating the same $\widehat{\Sigma}$. An interesting case is when one Σ in the class is convex-compact-valued, we have encountered this case in section 3 where the normal cones $N(x)$ to the level sets are locally generated by the compact convex sets $\Gamma_e(x)$. This map Γ_e is cyclically pseudomonotone and presents good continuity properties. It could be considered as a valuable substitute for the subdifferentials of convex functions if one essential property was not missing. Recall that if f is a proper closed convex function and f^* is its conjugate, then one has the equivalence

$$x^* \in \partial f(x) \Longleftrightarrow x \in \partial f^*(x^*).$$

A similar duality exists also on monotone maps. The point-to-set map map Σ is (cyclically) monotone if and only if its inverse Σ^{-1} is so. Actually, monotonicity is more a concept on graphs than on maps. Thus by definition $G \subset \mathbb{R}^n \times \mathbb{R}^n$ is said to be monotone if and only if

$$(x, x^*), (y, y^*) \in G \Longrightarrow \langle x^* - y^*, x - y \rangle \geq 0.$$

which corresponds to the monotonicity of both maps Σ and Σ^{-1} defined by

$$\Sigma(x) = \{\, x^* \,:\, (x, x^*) \in G \,\}, \;\; \Sigma^{-1}(x^*) = \{\, x \,:\, (x, x^*) \in G \,\}.$$

Our next goal is to define a similar duality for pseudoconvexity and pseudomonotonicity.

5 Duality in generalised monotonicity

Prior to the introduction of this new concept of pseudomonotonicity and in order to show its applicability, we provide two examples. The first concerns the normal cones to the level sets of a function.

The proof of the following result is the same as the proof of proposition 3.1.

Proposition 5.1 *Assume that f is g-pseudoconvex. Assume that $a \in \mathrm{int}\,(\mathrm{dom}\,(\mathrm{f}))$ is such that there exists λ with $\mathrm{int}\,(\mathrm{S}_\lambda) \neq \emptyset$ and $a \notin \mathrm{cl}\,(\mathrm{S}_\lambda)$. Given $b \in \mathrm{int}\,(\mathrm{S}_\lambda)$ there exist an open nonempty convex cone D and an open convex neighbourhood $V \subseteq \mathrm{dom}\,(\mathrm{f}) \cap (\mathrm{b} + \mathrm{D})$ of a such that $f(y) < f(x)$ whenever $x, y \in V$ with $x - y \in D$. Moreover, $\langle x^*, d \rangle > 0$ whenever $d \in D$, $0 \neq x^* \in N(x)$ and $x \in V$.*

With the assumptions of the proposition, for any closed convex cone $K \subseteq D$ with nonempty interior, one has $f(y) < f(x)$ whenever $x, y \in V$ with $x - y \in K$, $x \neq y$. We choose K in such a way that $\mathrm{int}\,(\mathrm{K}^+)$ is nonempty. Then we take for V an open convex neighbourhood of a such that $V \subseteq \mathrm{dom}\,(\mathrm{f}) \cap (\mathrm{b} + \mathrm{K})$. Then one has again $f(y) < f(x)$ whenever $x, y \in V$ with $x - y \in K$, $x \neq y$ and therefore

$$x^* \in \mathrm{int}\,(\mathrm{K}^+) \;\; \text{whenever} \;\; 0 \neq \mathrm{x}^* \in \mathrm{N(x)} \;\; \text{and} \;\; \mathrm{x} \in \mathrm{V}.$$

We define the point-to-set map Γ on V by

$$\Gamma(x) = \{x^* \in \mathbb{R}^n \,:\, x^* \in N(x),\; \langle x^*, x - b \rangle = 1\}.$$

The second example is on maps which are not necessarily associated with a function. Let us consider $\Sigma : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ and $a \in \mathrm{int}\,(\mathrm{dom}\,(\Sigma))$. Assume that there exist a neighbourhood V of a and a convex compact set A such that $0 \notin A \supseteq \Sigma(x)$ for all $x \in V$. Let K be a closed convex cone with nonempty interior such that $\mathrm{int}\,(\mathrm{K}^+) \supset \mathrm{A}$, such a cone exists. Next, take $b \in \mathbb{R}^n$ such that $V \subset b + K$, such a b exists. Then,

$$x^* \in \mathrm{int}\,(\mathrm{K}^+) \ \text{ whenever } \ \mathrm{x}^* \in \Sigma(\mathrm{x}) \ \text{ and } \ \mathrm{x} \in \mathrm{V}.$$

We define the point-to-set map Γ on V by

$$\Gamma(x) = \{x^* \in \mathbb{R}^n \,:\, x^* \in \Sigma(x),\ \langle x^*, x - b\rangle = 1\}.$$

In both cases we can proceed to a change in the origin so that we have $b = 0$.

According to these examples, given $K \subset \mathbb{R}^n$ a closed convex cone such that $\mathrm{int}\,(\mathrm{K})$ and $\mathrm{int}\,(\mathrm{K}^+)$ are nonempty, we define the set

$$E = \{(x, x^*) \in K \times K^+ \,:\, \langle x^*, x\rangle = 1\}.$$

Next, we say that a point-to-set map Γ with $\mathrm{graph}\,(\Gamma) \subset \mathrm{E}$ is:

- *D-cyclically quasimonotone of order p* (with $p \geq 1$) if for all ordered family $\{(x_i, x_i^*)\}_{i=0,\cdots,p+1} \subset \mathrm{graph}\,(\Gamma)$ with $(x_0, x_0^*) = (x_{p+1}, x_{p+1}^*)$ one has

$$\min_{i=0,\cdots,p} [\,\langle x_i^*, x_{i+1}\rangle\,] \leq 1,$$

- *D-cyclically pseudomonotone of order p* (with $p \geq 1$) if for all ordered family $\{(x_i, x_i^*)\}_{i=0,\cdots,p+1} \subset \mathrm{graph}\,(\Gamma)$ with $(x_0, x_0^*) = (x_{p+1}, x_{p+1}^*)$ one has

$$\min_{i=0,\cdots,p} [\,\langle x_i^*, x_{i+1}\rangle\,] \leq 1,$$

  and, in case where the minimum is 1, $\langle x_i^*, x_{i+1}\rangle = 1$ for $i = 0, 1, \cdots, p$,

- Γ is said to be *D-quasimonotone* (*D-pseudomonotone*) if D-cyclically quasimonotone (D-cyclically pseudomonotone) of order 1,

- Γ is said to be *D-cyclically quasimonotone* (*D-cyclically pseudomonotone*) if D-cyclically quasimonotone (D-cyclically pseudomonotone) of any order p.

Denote by G and Γ^{-1} the graph and the inverse of Γ, i.e.,

$$G = \{(x, x^*) \,:\, x^* \in \Gamma(x)\} = \{(x, x^*) \,:\, x \in \Gamma^{-1}(x^*)\}.$$

It is obvious that Γ is D-quasimonotone (D-pseudomonotone) if and only if Γ^{-1} is so, the following result is just a little less obvious.

Proposition 5.2 *If Γ is D-cyclically quasimonotone (D-cyclically pseudomonotone) of order p, so is Γ^{-1}.*

Proof: Given $(x_0, x_0^*), (x_1, x_1^*), \cdots, (x_{p+1}, x_{p+1}^*) \in G$ with $(x_0, x_0^*) = (x_{p+1}, x_{p+1}^*)$, we must prove that

$$\min_{i=0,\cdots,p} [\, \langle x_{i+1}^*, x_i \rangle \,] \leq 1,$$

and, for D-cyclically pseudomonotonicity, in case where the minimum is 1, $\langle x_{i+1}^*, x_i \rangle = 1$ for $i = 0, \cdots, p$. For $i = 0, 1, \cdots, p+1$ set $j = p+1-i$ and $(y_j, y_j^*) = (x_i, x_i^*)$. Then, $(y_0, y_0^*) = (y_{p+1}, y_{p+1}^*)$ and, since Γ is D-cyclically quasimonotone, one has

$$\min_{i=0,1,\cdots,p} [\, \langle x_{i+1}^*, x_i \rangle \,] = \min_{j=p+1,p,\cdots,1} [\, \langle y_{j-1}^*, y_j \rangle \,] = \min_{k=0,1,\cdots,p} [\, \langle y_k^*, y_{k+1} \rangle \,] \leq 1.$$

If Γ is D-cyclically pseudomonotone and

$$1 = \min_{i=0,1,\cdots,p} [\, \langle x_{i+1}^*, x_i \rangle \,] = \min_{k=0,1,\cdots,p} [\, \langle y_k^*, y_{k+1} \rangle \,],$$

then $\langle y_k^*, y_{k+1} \rangle = 1$ for all k and thereby $\langle x_{i+1}^*, x_i \rangle = 1$ for all i. ∎

The letter "D" in the notation is set in order to set in evidence the duality existing between the map and its inverse. As for monotonicity and cyclic monotonicity, D-(cyclic) quasi/pseudomonotonicity applies to the graph.

The duality existing between a monotone point-to-set map Γ and its inverse Γ^{-1} corresponds to the duality between a convex function f and its Fenchel-conjugate f^*. Such a correspondence exists between generalised monotone maps and generalised convex functions. Actually, this correspondence is at the origin of our definition of D-generalised monotonicity. It is related to the consumer theory. We describe it briefly.

Let us consider the situation where the behaviour of a consumer can be described through a utility function u: the consumer problem consists in maximizing the utility on the commodity set G (here, we assume that $G = \mathbb{R}_+^n$) subject to a budget

constraint. Let $\hat{p} \in \mathbb{R}^n_+,\ \hat{p} \neq 0$ be the vector of prices and $w > 0$ be the income of the consumer. Then the problem of the consumer is

$$\text{maximize } u(x) \text{ subject to } x \geq 0 \text{ and } \hat{p}^t x \leq w.$$

Set $p = w^{-1}\hat{p}$, then the problem reduces to

$$v(p) = \max [\ u(x)\ :\ x \geq 0,\ p^t x \leq 1].$$

Denote by $X(p)$ the set of optimal solutions of this problem. The multivalued map X is called the *demand correspondence*, the function v the *indirect utility function* associated to u. Next, define

$$\hat{u}(x) = \min [\ v(p)\ :\ p \geq 0,\ p^t x \leq 1].$$

Denote by $P(x)$ the set of optimal solutions of this problem.

Under reasonable economics considerations, the function u is increasing and quasiconcave and coincides with $\hat{u}$, v is decreasing and quasiconvex. Moreover, $X(p)$ generates the normal cone at p to the lower level set of v and $P(x)$ generates the normal cone at x to the upper level set of u. Finally,

$$p \in P(x) \Longleftrightarrow [u(x) = v(p) \text{ and } \langle p, x\rangle = 1] \Longleftrightarrow x \in X(p).$$

Thus the duality in between a convex function f and its conjugate f^* (a monotone map Γ and its inverse Γ^{-1}) is transposed to the duality between the direct utility function u and its indirect utility function v (the demand map X and its inverse P). X and P enjoy D-cyclic generalised monotonicity properties.

6 Maximality

Maximality is of a major importance in the study of monotone maps. In the same spirit one says that $G \subset E$ is *maximal D-quasimonotone* if for any D-quasimonotone subset S of E which contains G, one has $S = G$. In order to analyse the maximality of $G \subset E$, we introduce the set

$$\widetilde{G} = \{(x, x^*) \in E\ :\ \min [\ \langle x^*, y\rangle, \langle y^*, x\rangle\ ] \leq 1\ \ \forall\, (y, y^*) \in G\}. \tag{15}$$

We list below some rather obvious properties of $\widetilde{G}$.

1. G is D-quasimonotone if and only if $G \subset \widetilde{G}$.

2. $G_1 \subset G_2$ implies $\widetilde{G}_1 \supset \widetilde{G}_2$.

3. The set $\widetilde{G}$ is closed.

4. If G is D-quasimonotone and $(x, x^*) \in \widetilde{G}$, then $G \cup \{(x, x^*)\}$ is D-quasimonotone.

5. If G is D-quasimonotone, then $(x, x^*) \in \widetilde{G}$ if and only if $G \cup \{(x, x^*)\}$ is D-quasimonotone.

6. G is maximal D-quasimonotone if and only if $G = \widetilde{G}$.

Let $\widetilde{\Gamma} : K \rightrightarrows K^+$ be the point-to-set map with graph $\widetilde{G}$. Then, for all $x \in K$,

$$\widetilde{\Gamma}(x) = \bigcap_{(y,y^*) \in Y(x)} \{x^* \in K^+ \,:\, \langle x^*, y\rangle \leq 1,\, \langle x^*, x\rangle = 1\}, \tag{16}$$

where

$$Y(x) = \{(y, y^*) \in G \,:\, \langle y^*, x\rangle > 1\}.$$

Let us define,

$$T(x) = \{x^* \in K^+ \,:\, \langle x^*, x\rangle = 1\}.$$

By construction, $T(x)$ is convex and closed and $\widetilde{\Gamma}(x) \subseteq T(x)$. Moreover, in case where $x \in \operatorname{int}(\mathrm{K})$, there exist a compact set T and a neighbourhood V of x such that $T(y) \subset T$ for all $y \in V$. The following properties of $\widetilde{\Gamma}$ are easily obtained:

1. $\widetilde{\Gamma}(x)$ is a (may be empty) closed convex set.

2. If G is D-quasimonotone then $G \cup [\{x\} \times \widetilde{\Gamma}(x)]$ is D-quasimonotone.

3. If $x \in \operatorname{int}(\mathrm{K})$, then $\widetilde{\Gamma}(y)$ is locally bounded in a neighbourhood of x.

4. If $x \in \operatorname{int}(\mathrm{K})$, then, for all open set $\Omega \supset \widetilde{\Gamma}(x)$, there exists a neighbourhood V of x such that $\widetilde{\Gamma}(V) \subset \Omega$.

Proposition 6.1 *Assume that Γ is D-quasimonotone and $x \in \operatorname{int}(\mathrm{K})$. Then $\widetilde{\Gamma}(x)$ is nonempty.*

Proof: For simplicity, we give the proof when $K = \mathbb{R}^n_+$, the proof of the general case being easily deduced from the special case. Assume for contradiction that $\widetilde{\Gamma}(x) = \emptyset$. In view of (16) and because $T(x)$ is compact, $p \in \mathbb{N}$ and $(y_i, y_i^*) \in \text{graph}\,(\Gamma)$, i $= 1, \cdots,$ p, exist so that $\langle y_i^*, x - y_i\rangle > 0$ for all i and

$$\widetilde{\Gamma}(x) = \bigcap_{i=1,\cdots,p} \{x^* \,:\, x^* \geq 0,\ \langle x^*, x - y_i\rangle \geq 0,\ \langle x^*, x\rangle = 1\}.$$

Or equivalently,

$$+\infty = \inf_{x^*} [\,\langle x^*, 0\rangle \,:\, x^* \geq 0,\ Yx^* \geq 0,\ \langle x^*, x\rangle = 1\,],$$

where Y is the $p \times n$ matrix with lines $(x - y_i)^t$. By duality in linear programming,

$$+\infty = \sup_{t,u} [\,t + \langle 0, u\rangle \,:\, u \geq 0,\ tx + Y^t u \leq 0\,],$$

which means that there is $t > 0$ and $u \geq 0$ such that

$$(t + \sum_i u_i)x \leq \sum_i u_i y_i.$$

Since $x > 0$, $u = 0$ is not possible, so that we can assume that $\sum u_i = 1$. Then the condition means there is at least one i such that $x < y_i$. This is not possible since $y_i^* \geq 0$ and $\langle y_i^*, x - y_i\rangle > 0$. ∎

Corollary 6.1 *Assume that* Γ *is maximal D-quasimonotone. Then,* $\text{dom}\,(\Gamma) \subseteq \text{int}\,(\text{K})$.

Proof: If Γ is maximal D-quasimonotone, Γ coincides with $\widetilde{\Gamma}$. ∎

Given $S \subseteq \text{dom}\,(\Gamma)$, we say that $a^* \in \Gamma^l_S(a)$ if there exists a sequence $\{(x_k, x_k^*)\}_k$ contained in $\text{graph}\,(\Gamma) \cap (\text{S} \times \text{K}^+)$ which converges to (a, a^*). Next, we denote by $\Gamma^c_S(a)$ the closed convex hull of $\Gamma^l_S(a)$. When S coincides with $\text{dom}\,(\Gamma)$ we set $\Gamma^c(a) = \Gamma^c_S(a)$ and $\Gamma^l(a) = \Gamma^l_S(a)$.

Proposition 6.2 *Assume that* Γ *is D-quasimonotone and* $a \in K$. *Then the inclusion* $\Gamma^c_S(a) \subseteq \widetilde{\Gamma}(a)$ *holds. Moreover the maps* Γ^l_S *and* Γ^c_S *are D-quasimonotone.*

Proof: The graph of $\widetilde{\Gamma}$ is closed and contains the graph of Γ. Hence $\Gamma^l_S(a) \subseteq \widetilde{\Gamma}(a)$. Since $\widetilde{\Gamma}(a)$ is closed and convex, it contains the closed convex hull of $\Gamma^l_S(a)$.

Let $(x,x^*),(y,y^*) \in \mathrm{graph}\,(\Gamma^{\mathrm{l}}_{\mathrm{S}})$. We must prove that $\min(\langle x^*,y\rangle,\langle y^*,x\rangle) \leq 1$. Let $\{(x_k,x_k^*)\}_k,\{(y_k,y_k^*)\}_k \subset \mathrm{graph}\,(\Gamma) \cap (\mathrm{S}\times \mathrm{K}^+)$ be two sequences converging to (x,x^*) and (y,y^*) respectively. Then $\min(\langle x_k^*,y_k\rangle,\langle y_k^*,x_k\rangle) \leq 1$. Pass to the limit when $k \to +\infty$.

Next, let $(x,x^*),(y,y^*) \in \mathrm{graph}\,(\Gamma^{\mathrm{c}}_{\mathrm{S}})$. There exist two positive integers p and q, $t_i,s_j>0$, $x_i^*,y_j^* \in K^+$, $i=1,\cdots,p$, $j=1,\cdots,q$ such that

$$(x,x_i^*) \in \mathrm{graph}\,(\Gamma^{\mathrm{l}}_{\mathrm{S}}),\ \ (\mathrm{y},\mathrm{y}^*_{\mathrm{j}}) \in \mathrm{graph}\,(\Gamma^{\mathrm{l}}_{\mathrm{S}}),\quad \forall\,\mathrm{i}=1,\cdots,\mathrm{p},\ \forall\,\mathrm{j}=1,\cdots,\mathrm{q}$$

$$x^*=\sum_{i=1}^{p} t_i x_i^*,\ y^*=\sum_{j=1}^{q} s_j y_j^*,\quad 1=\sum_{i=1}^{p} t_i=\sum_{j=1}^{q} s_j.$$

Assume for instance that $\langle x^*,y\rangle>1$. Then there is $i\in\{1,\cdots,p\}$ such that $\langle x_i^*,y\rangle>1$. Next, since Γ^l_S is D-quasimonotone and $x_i^* \in \Gamma^l_S(x)$, one has $\langle y^*,x\rangle \leq 1$ for all $y^* \in \Gamma^l_S(y)$. In particular $\langle y_j^*,x\rangle \leq 1$ for all j and $\langle y^*,x\rangle=\sum_{j=1}^q s_j\langle y_j^*,x\rangle \leq 1$. The opposing case where $\langle y^*,x\rangle>1$ is treated in the same way. ■

Theorem 6.1 *Assume that* Γ *is D-quasimonotone,* $a \in \mathrm{int}\,(\mathrm{K})$ *and* $S \subseteq \mathrm{dom}\,(\Gamma)$ *is such that there exists a neighbourhood* V *of* a *such that* $\mathrm{cl}\,(\mathrm{V}\cap\mathrm{S})=\mathrm{cl}\,(\mathrm{V})$. *Then,* $\Gamma^c_S(a)=\widetilde{\Gamma}(a)$.

Proof: Consider any sequence $\{x_k\} \subset S$ converging to a. Let $x_k^* \in \Gamma(x_k)$, the sequence $\{x_k^*\}$ is bounded and therefore has cluster points. These cluster points are in $\Gamma^l_S(a)$. It follows that $\Gamma^l_S(a)$ is nonempty.

If $\widetilde{\Gamma}(a)$ is a singleton, the equality $\Gamma^c_S(a)=\widetilde{\Gamma}(a)$ necessarily holds. In the opposite case, it is enough to prove that any extremal point a^* of the convex set $\widetilde{\Gamma}(a)$ is actually in $\Gamma^l_S(a)$. There exists $d\in\mathbb{R}^n$ such that $\langle b^*-a^*,d\rangle<0$ for all $b^*\in\widetilde{\Gamma}(a)$, $b^*\neq a^*$. Since $\langle a^*,a\rangle=1$, there exists $\alpha\in\mathbb{R}$ such that $\langle a^*,d+\alpha a\rangle=0$. Next, since $\langle a^*,a\rangle=\langle b^*,a\rangle=1$, we have

$$\langle b^*,d+\alpha a\rangle=\langle b^*-a^*,d+\alpha a\rangle=\langle b^*-a^*,d\rangle<0\quad \forall\, b^*\in\widetilde{\Gamma}(a),\ b^*\neq a^*. \tag{17}$$

Set $x_k=a+\frac{1}{k}(d+\alpha a)$, then $\langle a^*,x_k-a\rangle=0$. Next, take $z_k\in S$ such that $\langle a^*,z_k-a\rangle>0$ and $k^2\|x_k-z_k\|\leq 1$. Such a z_k exists. Let $z_k^*\in\Gamma(z_k)$. Since Γ is D-quasimonotone one has

$$0\leq k\langle z_k^*,z_k-a\rangle=k\langle z_k^*,z_k-x_k\rangle+\langle z_k^*,d+\alpha a\rangle.$$

Let z^* be a cluster point of the bounded sequence $\{z_k^*\}$, then $z^* \in \Gamma_S^l(a)$. Passing to the limit, one obtains $\langle z^*, d + \alpha a\rangle \geq 0$. It follows from (17) that $z^* = a^*$. ■

The following corollaries are immediate consequences of the theorem.

Corollary 6.2 *Assume that* Γ *is D-quasimonotone and* $S \subseteq \mathrm{dom}\,(\Gamma)$ *is such that* $\mathrm{cl}\,(\mathrm{S}) = \mathrm{K}$. *Then,* Γ_S^c *is the unique maximal D-quasimonotone point-to-set map containing* Γ *on* $\mathrm{int}\,(\mathrm{K})$.

Corollary 6.3 *Assume that* Γ *is D-quasimonotone and* $\mathrm{int}\,(\mathrm{K}) \subseteq \mathrm{dom}\,(\Gamma)$. *Then,* Γ *is maximal D-quasimonotone on* $\mathrm{int}\,(\mathrm{K})$ *if and only if* $\Gamma^c(x) = \Gamma(x)$ *for all* $x \in \mathrm{int}\,(\mathrm{K})$.

Next, we turn our interest in maximal cyclic quasiconvexity. In connection with property 5 of $\widetilde{G}$, given $G \subset E$, we introduce the set

$$\widetilde{G}_c = \{(x, x^*) \in E \,:\, G \cup \{(x, x^*)\} \text{ is D-cyclically quasimonotone}\}.$$

Clearly, $\widetilde{G}_c \subseteq \widetilde{G}$, G is D-cyclically quasimonotone if and only if $\widetilde{G}_c$ is so and G is maximal D-cyclically quasimonotone if and only if $G = \widetilde{G}_c$.

Thus, given $G \subset E$, one has

$$\widetilde{G}_c = \bigcap_{J \in \mathcal{J}} \{(x, x^*) \in E \,:\, \min[\,\langle x^*, a_1\rangle, \langle a_p^*, x\rangle\,] \leq 1\,\}, \tag{18}$$

where $\mathcal{J}$ is the family of ordered finite subsets $J = \{(a_i, a_i^*)\}_{i=1,\cdots,p}$ of G such that $\langle a_i^*, a_{i+1}\rangle > 1$ for $i = 1, \cdots, p-1$.

Denote by $\widetilde{\Gamma}_c : K \rightrightarrows K^+$ be the point-to-set map with graph $\widetilde{G}_c$. Then, for all $x \in K$,

$$\widetilde{\Gamma}_c(x) = \bigcap_{J \in \mathcal{J}(x)} \{x^* \in K^+ \,:\, \langle x^*, a_1\rangle \leq 1,\, \langle x^*, x\rangle = 1\,\}, \tag{19}$$

where $\mathcal{J}(x)$ is the family of ordered finite subsets $J = \{(a_i, a_i^*)\}_{i=1,\cdots,p}$ of G such that $\langle a_i^*, a_{i+1}\rangle > 1$ for $i = 1, \cdots, p-1$ and $\langle a_p^*, x\rangle > 1$.

The following properties are direct consequences of (18) and (19).

1. $\widetilde{G}_c$ is closed.
2. For all $x \in K$, $\widetilde{\Gamma}_c(x) \subseteq \widetilde{\Gamma}(x) \subseteq T(x)$ for all $x \in K$.

3. for all $x \in K$, $\widetilde{\Gamma}_c(x)$ is a (may be empty) closed convex set.

4. If G is D-cyclically quasimonotone then $G \cup [\{x\} \times \widetilde{\Gamma}(x)]$ is D-cyclically quasimonotone.

5. If $x \in \text{int}\,(\text{K})$, then $\widetilde{\Gamma}_c(y)$ is locally bounded in a neighbourhood of x.

6. If $x \in \text{int}\,(\text{K})$, then, for all open set $\Omega \supset \widetilde{\Gamma}_c(x)$, there exists a neighbourhood V of x such that $\widetilde{\Gamma}_c(V) \subset \Omega$.

One says that $G \subset E$ is *maximal D-cyclically quasimonotone* if for any D-cyclically quasimonotone subset S of E which contains G, one has $S = G$.

Proposition 6.3 *Assume that G is D-cyclically quasimonotone and maximal D-quasimonotone. Then G is maximal D-cyclically quasimonotone.*

Proof: On one hand one has $G \subseteq \widetilde{G}_c \subseteq \widetilde{G}$ and on the other hand $G = \widetilde{G}$. It follows that $\widetilde{G}_c = G$. ■

The following result is the main tool of the results given later.

Theorem 6.2 ***[Afriat]*** *Assume that $A = \{(a_i, a_i^*),\, i = 0, 1, \cdots q\}$ is a finite D-cyclically pseudomonotone subset of E. Then, there exist $\mu_i \in \mathbb{R}$ and $\lambda_i > 0$, $i = 0, 1, \cdots, q$, such that for the piecewise linear convex function*

$$f(x) = \max_{i=0,1,\cdots,q} [\, \mu_i + \lambda_i \langle a_i^*, x - a_i \rangle \,]$$

one has $f(x_i) = \mu_i$ for all i. It follows that the domain of the conjugate f^* of f is the convex hull of the points $\lambda_i a_i^*$, $i = 0, 1, \cdots, q$.

Theorem 6.3 *Assume that Γ is D-cyclically pseudomonotone with $\text{dom}\,(\Gamma) \neq \emptyset$. Then $\Gamma_c(x) \neq \emptyset$ for all $x \in \text{int}\,(\text{K})$. If in addition $\Gamma(y) \subset \text{int}\,(\text{K}^+)$ for all $y \in \text{dom}\,(\Gamma)$, then $\widetilde{\Gamma}_c(x) \cap \text{int}\,(\text{K}^+) \neq \emptyset$ for all $x \in \text{int}\,(\text{K})$.*

Proof: Assume for contradiction that $x \in \text{int}\,(\text{K})$ with $\widetilde{\Gamma}_c(x) = \emptyset$. The set $T(x)$ is compact and therefore, in view of (19), $\widetilde{\Gamma}_c(x)$ is an intersection of compact sets. Hence, there exist q integer and, for $k = 1, \cdots, q$, $J_k(x) = \{(a_{ik}, a_{ik}^*)\}_{i=1,\cdots,p_k} \in \mathcal{J}(x)$ such that

$$\emptyset = \widetilde{\Gamma}_c(x) = \bigcap_{k=1,\cdots,q} \{x^* \in K^+ \,:\, \langle x^*, a_{1k} \rangle \leq 1,\, \langle x^*, x \rangle = 1\,\}. \tag{20}$$

Let A be the family of all points (a_{ik}, a_{ik}^*). Let f, μ_{ik}, λ_{ik} as in theorem 6.2. The function f is convex piecewise linear. Hence, for any $y \in \mathbb{R}^n$, $\partial f(y)$ is nonempty and contained in the convex hull of the points $\lambda_{ik} a_{ik}^*$. We deduce that $0 \notin \partial f(y) \subset K^+$. In particular, there exist $x^* \in K^+$ and $\lambda > 0$ such that $\lambda x^* \in \partial f(x)$ and $\lambda \langle x^*, x \rangle = 1$. The set $A \cup \{(x, x^*)\}$ is D-cyclically pseudomonotone because ∂f is monotone. In view of (20), there is $k \in \{1, \cdots, q\}$ such that $\langle x^*, a_{1k} \rangle > 1$. Take $(a_{0k}, a_{0k}^*) = (x, x^*)$, then $\langle a_{i,k}^*, , a_{i+1,k}^* \rangle > 1$ for $i = 0, \cdots, p_k - 1$ and therefore $\langle a_{p_k k}^*, x \rangle < 1$, in contradiction with $J_k \in \mathcal{J}(x)$.

In case where $\Gamma(y) \subset \text{int}\,(\mathrm{K}^+)$ for all $y \in \text{dom}\,(\Gamma)$, then $a_{ik}^* \in \text{int}\,(\mathrm{K}^+)$ for all i, k. Hence x^* belongs also to $\text{int}\,(\mathrm{K}^+)$. ■

Corollary 6.4 *Assume that* Γ *is maximal D-cyclically quasimonotone and D-cyclically pseudomonotone. Then* $\text{dom}\,(\Gamma) \supset \text{int}\,(\mathrm{K})$.

Proof: If Γ is maximal D-cyclically quasimonotone then $\widetilde{\Gamma}_c = \Gamma$. ■

Combining the above results with theorem 3.1, we see that the normal cone map associated to a g-pseudoconvex function is maximal quasimonotone and cyclically pseudomonotone.

7 Historical comments and references

Quasiconvex functions have been introduced by De Finetti [13], the motivation was in the modelling of some economic problems, if one can reasonably consider quasiconcave utility functions, concavity of these functions is a too strong assumptions. Economics is still one of the main applications of generalised convexity, mainly in consumer and production theories. Generalised monotonicity is also appeared first in economics as a property of the demand map issued from an utility maximisation problem. Thus the (weak, strong, general) axioms of revealed preferences ([17], [25]) correspond to what has been called after by mathematicians quasi/pseudomonotonicity and cyclic quasi/pseudomonotonicity. The revealed preferences problem consists in building an utility function from the knowledge of the demand correspondance. It is still a problem of actuality [15], [11], [14]. Theorem 6.2 is due to Afriat [2].

The introduction of pseudoconvex functions under the form given in (6) is due to Ponstein [23]. The introduction of pseudomonotone maps to Karamardian [22].

Since an important literature has been devoted to quasi/pseudoconvex functions and quasi/pseudomonotone maps. One recent textbook on generalised convex functions and monotone maps is the reference [16]. In particular, chapter 2 [9] is concerned with characterizations of generalized convex functions and generalised monotone maps and chapter 3 [10] with continuity of generalized convex functions.

Borde and Crouzeix [5] were the first to consider normal cones in quasiconvex analysis and consider what we call in this paper d-pseudoconvex functions. In particular, they study the continuity of these cones via convex compact generators as done in this paper. They have been followed by several authors [3], [4], [14].

Theorem 3.1 is due to Chabrillac and Crouzeix [6]. It is of major importance for the study of continuity properties of quasiconvex functions as shown in [10].

Duality between direct and indirect utility functions has been actively investigated. See, e.g., Roy [24], Lau [19], Diewert [12], Crouzeix [7, 8], Martinez-Legaz [20, 21].

Theorems 3.2 and 4.1 are the fruits of discussions between J.-P. Crouzeix and N. Hadjisavvas.

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